



Mark Scheme (Results)

January 2022

Pearson Edexcel International A Level
In Further Pure Mathematics F2 (WFM02)
Paper : WFM02/01

| Question | Scheme | Marks |
|-------------|--|--------------------------|
| 1(a) | $r = \sqrt{(-4)^2 + (-4\sqrt{3})^2} = \dots$ | M1 |
| | $\tan \theta = \frac{-4\sqrt{3}}{-4} \Rightarrow \theta = \tan^{-1}(\sqrt{3}) \pm \pi$ | M1 |
| | $8 \left(\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) \right)$ | A1 |
| | | (3) |
| (b) | $z = re^{i\theta} \Rightarrow (re^{i\theta})^3 = -4 - 4\sqrt{3}i \Rightarrow r^3 (e^{3i\theta}) = 8e^{-i\frac{2\pi}{3}}$ | |
| | $\Rightarrow r = \sqrt[3]{8} = 2$ | M1 |
| | $3\theta = -\frac{2\pi}{3} + 2k\pi \Rightarrow \theta = -\frac{2\pi}{9} + \left(\frac{2k\pi}{3}\right)$ | M1 |
| | $\text{So } z = 2e^{-\frac{8\pi i}{9}}, 2e^{-\frac{2\pi i}{9}}, 2e^{\frac{4\pi i}{9}}$ | A1ft A1 |
| | | (4) |

(7 marks)

Notes:

(a)

M1: For a correct attempt at the modulus, implied by a correct modulus if no method seen and allow recovery if correct answer follows a minor slip in notation.

M1: For an attempt to find a value of θ in the correct quadrant. Accept $\tan^{-1}(\sqrt{3}) \pm \pi$ or $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \pm \pi$

May be implied by sight of an of $-\frac{2}{3}\pi, \frac{4}{3}\pi, -\frac{5}{6}\pi, \frac{7}{6}\pi$.

A1: cao as in scheme, no other solution.

(b)

M1: Applies De Moivre's Theorem and proceeds to find a value for r ie $(\text{their } 8)^{\frac{1}{3}}$

M1: Proceeds to find at least one value for θ – ie their argument/3.

A1ft: At least two roots correct for their r and θ . (Must come from correct method, watch for correct roots coming from an incorrect angle due to errors.)

A1: All three correct roots and no others. Accept e.g $2e^{i\frac{8\pi}{9}}$ as a slip in notation, so allow marks.

| Question | Scheme | Marks |
|---|--|-------------|
| 2 | $2m^2 - 5m - 3 = 0 \Rightarrow (2m+1)(m-3) = 0 \Rightarrow m = \dots$ | M1 |
| | So C.F. is $(y_{CF} =) Ae^{\frac{1}{2}x} + Be^{3x}$ | A1 |
| | P.I. is $y_{PI} = axe^{3x}$ | B1 |
| | $\frac{dy_{PI}}{dx} = 3axe^{3x} + ae^{3x}, \frac{d^2y_{PI}}{dx^2} = 9axe^{3x} + 3ae^{3x} + 3ae^{3x}$ $\Rightarrow 2(9ax + 6a)e^{3x} - 5(3ax + a)e^{3x} - 3axe^{3x} = 2e^{3x} \Rightarrow a = \dots$ | M1 |
| | $a = \frac{2}{7}$ | A1 |
| | General solution is $y = Ae^{\frac{1}{2}x} + Be^{3x} + \frac{2}{7}xe^{3x}$ | B1ft |
| | | (6) |
| (6 marks) | | |
| Notes: | | |
| <p>M1: Forms and solves the auxiliary equation.</p> <p>A1: Correct complementary function (no need for $y = \dots$)</p> <p>B1: Correct form for the particular integral. Accept any PI that includes axe^{3x}, so e.g. $(ax+b)e^{3x}$ is fine.</p> <p>M1: Attempts to differentiate their PI twice and substitutes into the left hand side of the equation. The derivatives must be changed functions. There is no need to reach a value for the unknown(s) but their PI must contain an unknown constant.</p> <p>A1: Correct value of a (and any other coefficients as zero). Must have had a suitable PI</p> <p>B1ft: For $y =$ their CF + their PI. Must include the $y =$. The PI must be a function of x that matches their initial choice of PI, with their constants substituted.</p> | | |

| Question | Scheme | Marks |
|-------------------|--|-------------|
| 3(a) | Meet when $x^2 - 8x = \frac{4x}{4-x} \Rightarrow (x^2 - 8x)(4-x) = 4x \Rightarrow x(4x - 32 - x^2 + 8x - 4) = 0$ | M1 |
| | (so $x = 0$ or) $x^2 - 12x + 36 = 0$ | A1 |
| | $\Rightarrow x(x-6)^2 = 0 \Rightarrow x = \dots$ | M1 |
| | Meet at (6,-12) | A1 |
| | e.g. touch at (6,-12) as repeated root. | B1 |
| | | (5) |
| Alt | $\frac{d}{dx}(x^2 - 8x) = 2x - 8$ and $\frac{d}{dx}\left(\frac{4x}{4-x}\right) = \frac{4(4-x) - 4x(-1)}{(4-x)^2} = \frac{16}{(4-x)^2}$ | M1A1 |
| | $2x - 8 = \frac{16}{(4-x)^2} \Rightarrow (x-4)^3 = 8 \Rightarrow x = \dots$ | M1 |
| | Meet at (6,-12) | A1 |
| | e.g. $6^2 - 6 \times 9 = -12$ and $\frac{4 \times 6}{4-6} = -12$, so curves meet at tangent at (6,-12) | B1 |
| | | (5) |
| (b) | $x^2 - 8x = \frac{4x}{4+x} \Rightarrow x(x-8)(4+x) - 4x = 0 \Rightarrow x(x^2 - 4x - 36) = 0 \Rightarrow x = \dots$ | M1 |
| | $x = (0), 2 \pm 2\sqrt{10} \Rightarrow$ critical value is (0 and) $2 - 2\sqrt{10}$ | A1 |
| | Other C.V.'s are 0, ± 4 | B1 |
| | E.g. extremes are $x < 2 - 2\sqrt{10}$ and $x > 6$ or any two suitable ranges. | M1 |
| | Solution is $x < 2 - 2\sqrt{10}, -4 < x < 0, 4 < x < 6, x > 6$ | A1A1 |
| | | (6) |
| (11 marks) | | |

Notes:**(a)**

M1: Attempts to find intersection by setting equations equal and cross multiplies and factorises the x out or cancels.

A1: Correct quadratic reached. May be implied by solutions of 0,6 seen from the cubic (by calculator)

M1: Solves the quadratic to find roots.

A1: Obtains the correct point where the curves meet.

B1: Correct reason given for why the curves touch. Accepted “repeated root” as reason. As a minimum, accept “ $(x - 6)^2 = 0$ therefore touches”. Alternatively, accept discriminant = 0 shown with conclusion, or may find gradient at both points and show equal, with conclusion.

Alt:

M1: Attempts derivatives of both curves

A1: Both derivatives correct.

M1: Sets derivatives equal and solves to find x value where gradients agree.

A1: Obtains the correct point where the curves meet.

B1: Correct value checked in both curves with conclusion that they meet at a tangent or equivalent working as per main scheme.

(b)

M1: Attempts to find the intersection of the other branch of $\frac{4x}{4 - |x|}$ with $x^2 - 8x$. Allow for any attempt at

solving $\frac{4x}{4 + x} = x^2 - 8x$ that reaches a value for x

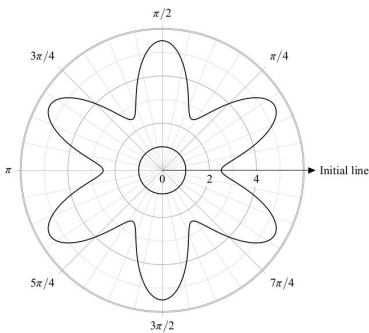
A1: Correct value of $2 - 2\sqrt{10}$ identified. (No need to see the second root rejected for this mark.)

B1: Both 0 and ± 4 identified as critical values for the ranges needed at some stage in working.

M1: Forms at least two suitable ranges from their critical values (allow if e.g. \leq is used instead of $<$). Likely to be the extreme ranges, so look for $x < 2 - 2\sqrt{10}$ and $x > 6$. However, allow if this latter is included as part of the range $x > 4$ for this mark.

A1: At least two correct ranges.

A1: Fully correct answer as in scheme.

| Question | Scheme | | Marks |
|--|--|---|-------|
| 4(a) |  | Completes to a closed loop with “petals” containing circle of radius 1 (whether the circle is drawn or not) | M1 |
| | | Fully correct – 6 petals in roughly the right places, but allow if curvature is not quite smooth. | A1 |
| | | Circle centre O radius 1. | B1 |
| | | | |
| | | | (3) |
| (b) | $\left(\frac{1}{2}\right) \int r^2 \, d\theta = \left(\frac{1}{2}\right) \int \left(16 - 12 \cos 6\theta + \frac{9}{4} \cos^2 6\theta\right) \, d\theta$ | M1 | |
| | $= \frac{1}{2} \int_0^{2\pi} \left(16 - 12 \cos 6\theta + \frac{9}{8} (1 + \cos 12\theta)\right) \, d\theta$ | M1 | |
| | $= \frac{1}{2} \left[16\theta - 2 \sin 6\theta + \frac{9}{8} \left(\theta + \frac{1}{12} \sin 12\theta \right) \right]_0^{2\pi}$ | M1 A1 | |
| | $A_{outer} = \frac{1}{2} \int_0^{2\pi} r^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} \left(16 - 12 \cos 6\theta + \frac{9}{4} \cos^2 6\theta\right) \, d\theta$ $= \frac{1}{2} \left(32\pi - 0 + \frac{9}{8} (2\pi + 0) - (0) \right)$ | dM1 | |
| | So Area required is $\frac{1}{2} \left(32\pi + \frac{9\pi}{4} \right) - \pi(1^2) = \dots$ | B1 | |
| | $= \frac{129}{8} \pi$ | A1 | |
| | | (7) | |
| (10 marks) | | | |
| Notes: | | | |
| <p>(a)</p> <p>M1: Allow for any closed loop that oscillates, though may not have the correct number of “petals” but require at least 4 . Need not have correct places of maximum radius.</p> <p>A1: Fully correct sketch, 6 “petals” in the right places, with maximum radius between the 5 and 6 radius lines, minimum between the 2 and 3 radius lines.</p> <p>B1: For a circle of radius 1 and centre O drawn.</p> <p>(b)</p> <p>M1: Attempts to square r as part of an integral for the outer curve, achieving a 3 term quadratic in $\cos 6\theta$</p> <p>M1: Applies the double angle formula to the \cos^2 term from their expansion (not dependent on the first M, but must have a \cos^2 term). Accept $\cos^2 6\theta \rightarrow \frac{1}{2}(\pm 1 \pm \cos 12\theta)$</p> <p>M1: Attempts to integrate, achieving the form $\alpha\theta + \beta \sin 6\theta + \gamma \sin 12\theta$ where $\alpha, \beta, \gamma \neq 0$</p> | | | |

A1: Correct integration – limits and the $\frac{1}{2}$ not needed. Look for $16\theta - 2\sin 6\theta + \frac{9}{8}\left(\theta + \frac{1}{12}\sin 12\theta\right)$ oe.

dM1: Depends on at least two of the previous M's being scored. For a correct overall strategy for the area contained in the outer loop, with an attempt at the r^2 (should be 3 term expansion). Correct appropriate limits and the $\frac{1}{2}$ should be present or implied by working, but note variations on the scheme are possible, e.g.

$2 \times \frac{1}{2} \int_0^\pi r^2 \, d\theta$, in which the $2 \times \frac{1}{2}$ may be implied rather than seen.

B1: Subtracts correct area of π for inner circle

A1: cso. Check carefully the integration was correct as the sin terms disappear with the limits.

| Question | Scheme | Marks |
|--|---|-----------|
| 5(a) | $\frac{dy}{dx} = \frac{1}{2}(4 + \ln x)^{-\frac{1}{2}} \times \frac{1}{x}$ | M1 A1 |
| | $\frac{d^2y}{dx^2} = \frac{1}{2} \frac{0 - \left(\sqrt{4 + \ln x} + x \times \frac{1}{2}(4 + \ln x)^{-\frac{1}{2}} \times \frac{1}{x} \right)}{x^2(4 + \ln x)}$ or $\frac{d^2y}{dx^2} = -\frac{1}{4x}(4 + \ln x)^{-\frac{3}{2}} \times \frac{1}{x} - \frac{1}{x^2} \times \frac{1}{2}(4 + \ln x)^{-\frac{1}{2}}$ oe | M1 |
| | $= \frac{\dots}{4x^2(4 + \ln x)^{\frac{3}{2}}} = -\frac{9 + 2 \ln x}{4x^2(4 + \ln x)^{\frac{3}{2}}}$ * | M1 A1* |
| | | (5) |
| | | |
| Alt(a) | $y^2 = 4 + \ln x \Rightarrow 2y \frac{dy}{dx} = \frac{1}{x}$ | M1 A1 |
| | $\Rightarrow 2y \frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 = -\frac{1}{x^2}$ | M1 |
| | $\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{2yx^2} - \frac{2}{8x^2y^3} = \frac{-2y^2 - 1}{4x^2y^3}$ | M1 |
| | $= -\frac{9 + 2 \ln x}{4x^2(4 + \ln x)^{\frac{3}{2}}}$ * | A1* |
| | | (5) |
| (b) | $y_{x=1} = 2, \left. \frac{dy}{dx} \right _{x=1} = \frac{1}{4}, \left. \frac{d^2y}{dx^2} \right _{x=1} = -\frac{9}{32}$ | M1 |
| | So $y = 2 + \frac{1}{4}(x-1) - \frac{1}{2!} \times \frac{9}{32}(x-1)^2 + \dots$ | M1 |
| | $= 2 + \frac{1}{4}(x-1) - \frac{9}{64}(x-1)^2 + \dots$ | A1 |
| | | (3) |
| (8 marks) | | |
| Notes: | | |
| (a) | | |
| M1: Attempts the derivative of y using the chain rule, look for $\frac{K}{x}(4 + \ln x)^{-\frac{1}{2}}$ oe | | |
| A1: Correct derivative. | | |
| M1: Attempts the second derivative of y using the product or quotient rule and chain rule. Look for the correct form for their $\frac{dy}{dx}$ for the answer up to slips in coefficients. | | |
| M1: Attempts to simplify to get correct denominator. Must be correct work for their second derivative, but may have been errors in differentiating. | | |
| A1*: For a correct unsimplified second derivative, with no errors before reaching the given answer. | | |
| Note it is a given answer so needs a suitable intermediate line with at least the formation of the correct common denominator between two fractions before reaching the answer. | | |

Alt:

M1: Squares and uses implicit differentiation to achieve $\alpha y \frac{dy}{dx} = \frac{\beta}{x}$

A1: Correct derivative.

M1: Differentiates again using implicit differentiation and product rule. Look for $\gamma y \frac{d^2 y}{dx^2} + \delta \left(\frac{dy}{dx} \right)^2 = \frac{\nu}{x^2}$

M1: Makes $\frac{d^2 y}{dx^2}$ the subject and forms single fraction with denominator $kx^2 y^3$

A1*: Obtains the correct second derivative, with no errors seen in working.

(b)

M1: Evaluates y , $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ at $x = 1$, if substitution is not seen, accept stated values for all three following attempts at the first and second derivatives as an attempt to find these.

M1: Applies Taylor's theorem with their values.

A1: Correct expression (don't be concerned if the $y =$ is missing.)

| | | |
|-----------------|---|------------|
| 5(b) Alt | $y = \sqrt{4 + \ln(1 + (x-1))} = \sqrt{4 + \left((x-1) - \frac{(x-1)^2}{2} + \dots \right)}$ | M1 |
| | $= 4^{\frac{1}{2}} + \frac{1}{2} \times 4^{-\frac{1}{2}} \times \left((x-1) - \frac{(x-1)^2}{2} \right) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!} \times 4^{-\frac{3}{2}} \times ((x-1) - \dots)^2 + \dots$ | M1 |
| | $= 2 + \frac{1}{4}(x-1) - \frac{1}{8}(x-1)^2 - \frac{1}{64}(x-1)^2 + \dots = 2 + \frac{1}{4}(x-1) - \frac{9}{64}(x-1)^2 + \dots$ | A1 |
| | | (3) |

Notes:

M1: Writes the x as $1 + (x-1)$ and attempts to expand using the Maclaurin series for $\ln(1+x)$ with correct expansion of $\ln(1+(x-1))$.

M1: Attempts a binomial expansion using their \ln expansion. Alternatively, may gain this before the first M

if they expand using \ln 's, e.g. $4^{\frac{1}{2}} + \frac{1}{2} 4^{-\frac{1}{2}} \ln x + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!} (\ln x)^2$

A1: Fully correct expression (don't be concerned if the $y =$ is missing.)

| Question | Scheme | Marks |
|-------------------|--|------------|
| 6(a) | Let $x = \arctan A$ and $y = \arctan B$ then $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$ Or $\tan(\arctan A - \arctan B) = \frac{\tan \arctan A - \tan \arctan B}{1 + \tan \arctan A \tan \arctan B}$ | B1 |
| | $\tan(x - y) = \frac{A - B}{1 + AB} \Rightarrow x - y = \arctan\left(\frac{A - B}{1 + AB}\right)$ | M1 |
| | So $\arctan A - \arctan B = x - y = \arctan\left(\frac{A - B}{1 + AB}\right)^*$ | A1* |
| | | (3) |
| (b) | $A = r + 2, B = r \Rightarrow \left(\frac{A - B}{1 + AB}\right) = \frac{r + 2 - r}{1 + (r + 2)r} = \frac{2}{...}$ | M1 |
| | $= \frac{2}{r^2 + 2r + 1} = \frac{2}{(1 + r)^2}^*$ | A1* |
| | | (2) |
| (c) | $\sum_{r=1}^n \arctan\left(\frac{2}{(1+r)^2}\right) = \sum_{r=1}^n (\arctan(r+2) - \arctan(r)) = ...$ | M1 |
| | $= (\cancel{\arctan 3} - \arctan 1) + (\cancel{\arctan 4} - \arctan 2) + (\cancel{\arctan 5} - \cancel{\arctan 3}) + ...$ $+ (\arctan(n+1) - \cancel{\arctan(n-1)}) + (\arctan(n+2) - \cancel{\arctan n})$ | A1 |
| | $= \arctan(n+2) + \arctan(n+1) - \arctan 2 - \arctan 1$ | M1 |
| | $= \arctan(n+2) + \arctan(n+1) - \arctan 2 - \frac{\pi}{4}$ | A1 |
| | | (4) |
| (d) | As $n \rightarrow \infty, \arctan n \rightarrow \frac{\pi}{2}$ | M1 |
| | So $\sum_{r=1}^{\infty} \arctan\left(\frac{2}{(1+r)^2}\right) = \frac{\pi}{2} + \frac{\pi}{2} - \arctan 2 - \frac{\pi}{4} = \frac{3\pi}{4} - \arctan 2$ | A1 |
| | | (2) |
| (11 marks) | | |

Notes:**(a)**

B1: For any correct statement or use of the compound angle formula with **consistent variables** of x and y or $\arctan A$ and $\arctan B$. Can be either way round (may be working in reverse).

M1: Attempts to apply \tan or \arctan on an appropriate identity with either x and y or $\arctan A$ and $\arctan B$.

Should have $\frac{\tan x \pm \tan y}{1 \pm \tan x \tan y}$ (oe with arctans or A 's and B 's) as part of the identity, and allow if they change between x, y and \arctan 's during the step.

A1*: Must have scored the B and M marks. Replaces $\tan x$ and $\tan y$ by A and B respectively if appropriate with fully correct work leading to the given result and conclusion made and no erroneous statements.

Note: for working in reverse e.g.

Let $x = \arctan A$ and $y = \arctan B$ then

$$\arctan A - \arctan B = \arctan\left(\frac{A-B}{1+AB}\right) \Leftrightarrow x - y = \arctan\left(\frac{A-B}{1+AB}\right) \Leftrightarrow \tan(x-y) = \frac{A-B}{1+AB} \quad \text{Scores M1}$$

$$\Leftrightarrow \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \quad \text{Scores B1 – but enter as the first mark.}$$

Which is the correct identity for $\tan(x-y)$ hence the result is true.

Score A1

The conclusion here must include reference to the identity being true, e.g. with a tick, or statement, before deducing the final result.

(b)

M1: Substitutes in $A = r + 2$ and $B = r$ and simplifies the numerator to 2 (may be implied)

A1*: Expands the denominator (must be seen) and then factorises to the given result, no errors seen.

(c)

M1: Applies the result of (a) to the series – allow if they have a different A and B due to error.

A1: At least first three and final two brackets of terms correctly written out – must be clear enough to show cancelling.

M1: Extracts the non-cancelling terms.

A1: Correct result with no errors seen – must see the $\arctan 1$ before reaching $\frac{\pi}{4}$.

Note: Insufficient terms to gain the first A is not an error, so M1A0M1A1 is possible if e.g. only the first two terms are shown. Condone missing brackets on $\arctan n + 1$ etc.

(d)

M1: Identifies the value \arctan tends towards as n increase. Need not see limits, as long as the value is identified.

A1: Correct answer.

| Question | Scheme | Marks |
|----------|---|-------|
| 7(a) | $z = (0+)iy \Rightarrow w = \frac{(1+i)iy + 2(1-i)}{iy-i} = \frac{-y+2+i(y-2)}{i(y-1)} = \frac{y-2+i(y-2)}{y-1}$ | M1 |
| | $\Rightarrow u = v \text{ or } \operatorname{Im} w = \operatorname{Re} w$ | A1 |
| | | (2) |
| (b) | $w = \frac{(1+i)z + 2(1-i)}{z-i} \Rightarrow z = \frac{2(1-i)+iw}{w-1-i} = \frac{2-v+i(u-2)}{u-1+i(v-1)}$ | M1 |
| | $\frac{2-v+i(u-2)}{u-1+i(v-1)} \times \frac{u-1-i(v-1)}{u-1-i(v-1)}$ $= \frac{(2-v)(u-1) + (u-2)(v-1) + i((u-1)(u-2) - (2-v)(v-1))}{\dots}$ $\operatorname{Im} z = 0 \Rightarrow (u-1)(u-2) - (2-v)(v-1) = 0$ | M1 |
| | $\Rightarrow (u-1)(u-2) - (2-v)(v-1) = 0 \Rightarrow u^2 - 3u + 2 + v^2 - 3v + 2 = 0$ | A1 |
| | $\Rightarrow \left(u - \frac{3}{2}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{1}{2}$ | M1 |
| | Centre is $\frac{3}{2} + \frac{3}{2}i$ and radius is $\frac{\sqrt{2}}{2}$ | A1A1 |
| | | (6) |

(8 marks)

Notes:

(a)

M1: Correct method to find the equation of the image line – e.g. substitutes in $z = iy$ and rearranges to Cartesian form. May use $x + iy$ and later set $x = 0$. Alternatively, may start as in (b) and then set $(2-v)(u-1) + (u-2)(v-1) = 0 \Rightarrow 2u - v - uv - 2 + uv + 2 - 2v - u = 0$ etc.

Another alternative is to find the image points of two points on the imaginary axis and to find the line from these.

A1: For $u = v$ or equation. Accept $\operatorname{Im} w = \operatorname{Re} w$, or $x = y$ if they have set $w = x + iy$.

(b)

Note: Accept work done in part (a) that is relevant to part (b) for credit if appropriate.

M1: Makes z the subject, substitutes $w = u + iv$ into the equation.

M1: Multiplies the numerator by the complex conjugate of denominator **and** extracts the imaginary part and sets it equal to zero to form an equation in u and v . Do not be concerned about the denominator.

A1: Correct equation in u and v for the circle.

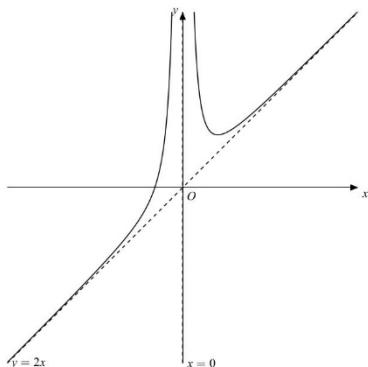
M1: Completes the square on their equation to extract centre and radius. Not dependent, so allow as long as a suitable equation in u and v has been reached.

A1: Correct centre or correct radius. Accept either $\frac{3}{2} + \frac{3}{2}i$ or $\left(\frac{3}{2}, \frac{3}{2}\right)$ for the centre.

A1: Correct centre and correct radius. As above. Accept equivalent forms (need not be simplified)

Allow the final two A marks if all that is wrong is an error in the denominator. (M1M0A0M1A1A1 is possible.)

| | | |
|---|--|--|
| 7(b) Alt1 | Real axis is $z = x(+0i)$, so $u + iv = \frac{(1+i)x + 2(1-i)}{x-i} = \frac{(1+i)x + 2(1-i)}{x-i} \times \frac{x+i}{x+i} =$ $\frac{(1+i)x^2 + 2x(1-i) + (i-1)x + 2(i+1)}{x^2+1} = \frac{x^2 + x + 2 + i(x^2 - x + 2)}{x^2+1}$ | M1 |
| | $u = \frac{x^2 + x + 2}{x^2 + 1} = 1 + \frac{x+1}{x^2+1}; v = \frac{x^2 - x + 2}{x^2 + 1} = 1 - \frac{x-1}{x^2+1} \Rightarrow u + v = 2 + \frac{2}{x^2+1}$ $\Rightarrow (u-1)^2 + (v-1)^2 = \frac{(x+1)^2 + (x-1)^2}{(x^2+1)^2} = \frac{2x^2+2}{(x^2+1)^2} = \frac{2}{x^2+1} = u + v - 2$ | M1 A1 |
| | $\Rightarrow \left(u - \frac{3}{2}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{1}{2}$ | M1 |
| | Centre is $\frac{3}{2} + \frac{3}{2}i$ and radius is $\frac{\sqrt{2}}{2}$ | A1A1 |
| | | (6) |
| <p>Notes</p> <p>M1: Sets $z = x$ in the equation (or uses $x + iy$ and later sets $y = 0$) and multiplies by complex conjugate.</p> <p>M1: Eliminates x from the equations (one suitable method is shown, others are possible).</p> <p>A1: Correct equation in u and v for the circle.</p> <p>M1: Completes the square on their equation to extract centre and radius</p> <p>A1: Correct centre or correct radius. Accept either $\frac{3}{2} + \frac{3}{2}i$ or $\left(\frac{3}{2}, \frac{3}{2}\right)$ for the centre.</p> <p>A1: Correct centre and correct radius. As above.</p> | | |
| 7(b) Alt 2 | <p style="text-align: center;">Unlikely to be seen</p> <p>As i and $-i$ are inverse points of the line, so their images are inverse points of the circle.</p> $i \rightarrow \infty, -i \rightarrow \frac{-i+1+2-2i}{-2i} = \frac{3}{2} + \frac{3}{2}i$ <p>Hence (as ∞ is the other point) the centre is $\frac{3}{2} + \frac{3}{2}i$</p> $0 \rightarrow \frac{2-2i}{-i} = 2+2i \quad \text{So radius is } \left \frac{3}{2} + \frac{3}{2}i - 2 - 2i \right = \dots$ $= \frac{\sqrt{2}}{2}$ | M1 M1 A1 M1 A1 A1 |
| (b) Alt 3 | <p>M1: Attempt to find images of three different points on the real axis.</p> <p>M1: Correct method to find centre from three points – e.g. intersection of two perpendicular bisectors.</p> <p>A1: Correct equation for the centre.</p> <p>M1: Uses centre and one point to find radius.</p> <p>A1: Correct centre</p> <p>A1: Correct radius</p> | |

| Question | Scheme | | Marks |
|------------|---|---|-------|
| 8(a) | $\frac{dv}{dx} = \frac{dy}{dx} - 2$ | | B1 |
| | $\frac{dy}{dx} + 2yx(y - 4x) = 2 - 8x^3 \rightarrow \frac{dv}{dx} + 2 + 2(v + 2x)x(v + 2x - 4x) = 2 - 8x^3$ | | |
| | $\rightarrow \frac{dv}{dx} + 2 + 2x(v^2 - 4x^2) = 2 - 8x^3$ | | M1 |
| | $\rightarrow \frac{dv}{dx} = -2xv^2 *$ | | A1* |
| | | | (4) |
| (b) | $\frac{1}{v^2} \frac{dv}{dx} = -2x \Rightarrow \int v^{-2} dv = -2 \int x dx$ | | B1 |
| | $\Rightarrow \frac{v^{-1}}{-1} = -2 \frac{x^2}{2} (+c)$ | | M1 |
| | $\Rightarrow \frac{1}{v} = x^2 + c$ | | A1 |
| | $\Rightarrow v = \frac{1}{x^2 + c}$ | | A1 |
| | | | (4) |
| (c) | $y = 2x + \frac{1}{x^2 + c}$ | | B1ft |
| | | | (1) |
| (d) | $-1 = 2 \times -1 + \frac{1}{1 + c} \Rightarrow c = \dots$ | | M1 |
| | $y = 2x + \frac{1}{x^2}$ | | A1 |
| |  | Attempts the sketch for their equation, with at least one of <ul style="list-style-type: none">- One branch correct- Vertical asymptote for their equation- Long term behaviour tends to infinity- Minimum in quadrant 1 | M1 |
| | | Fully correct shape, two branches tending to infinity as x tends to infinity both directions, with minimum in first quadrant No need for oblique asymptote marked. | A1 |
| | | y-axis a vertical asymptote labelled | B1ft |
| | | | (5) |
| (14 marks) | | | |
| Notes: | | | |
| (a) | | | |

B1: Correct differentiation of the given transformation. Allow any correct connecting derivative, e.g.

$$\frac{dy}{dv} = 1 + 2 \frac{dx}{dv} \text{ or } \frac{dv}{dy} = 1 - 2 \frac{dx}{dy}$$

M1: For a complete substitution into the equation (I).

M1: Applies difference of squares, or completely expands brackets of the left hand side. Alternatively, may rearrange and factorise to give $8x^2y - 2xy^2 - 8x^3 = -2x(y^2 - 4xy + 4x^2) = -2x(y - 2x)^2$ before substituting.

A1*: Reaches the given answer with no errors seen.

(b)

B1: Correct separation of the variables.

M1: Attempts the integration, usual rule, power increased by 1 on at least one term. No need for $+c$ for the method.

A1: Correct integration including the $+c$

A1: Correct expression for v .

(c)

B1: Follow through their answer to (b), so $y = 2x +$ their v from (b)

(d)

M1: Uses the point $(-1, -1)$ to find a value for the constant in their equation. Must have had a constant of integration in their equation to score this mark.

A1: Correct equation for y following a correct general solution. Withhold this mark for $y = 2x + \frac{1}{x^2} + c$ leading to the correct equation.

Note: the following three marks may be scored from a correct equation that arose from having no constant in

(b) or from $y = 2x + \frac{1}{x^2} + c$ (which gives the same equation).

M1: Attempts a sketch for their curve. See scheme. Look for at least one of the key features for their equation shown.

A1: Correct shape, two branches tending to infinity as x tends to infinity both directions with a minimum in first quadrant. Not a follow through mark, so must be the correct curve.

B1ft: Correct vertical asymptote at $x = 0$. Need not be labelled if it is clearly the y -axis. Follow through their equation as long as there is at least one vertical asymptote (ie for a negative c they need a pair of asymptotes symmetric about the y -axis).